Algorithms on Graphs

Umang Bhaskar & Juhi Chaudhary

Day 2, session 2: TSP

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STCS Vigyan Vidushi 2024

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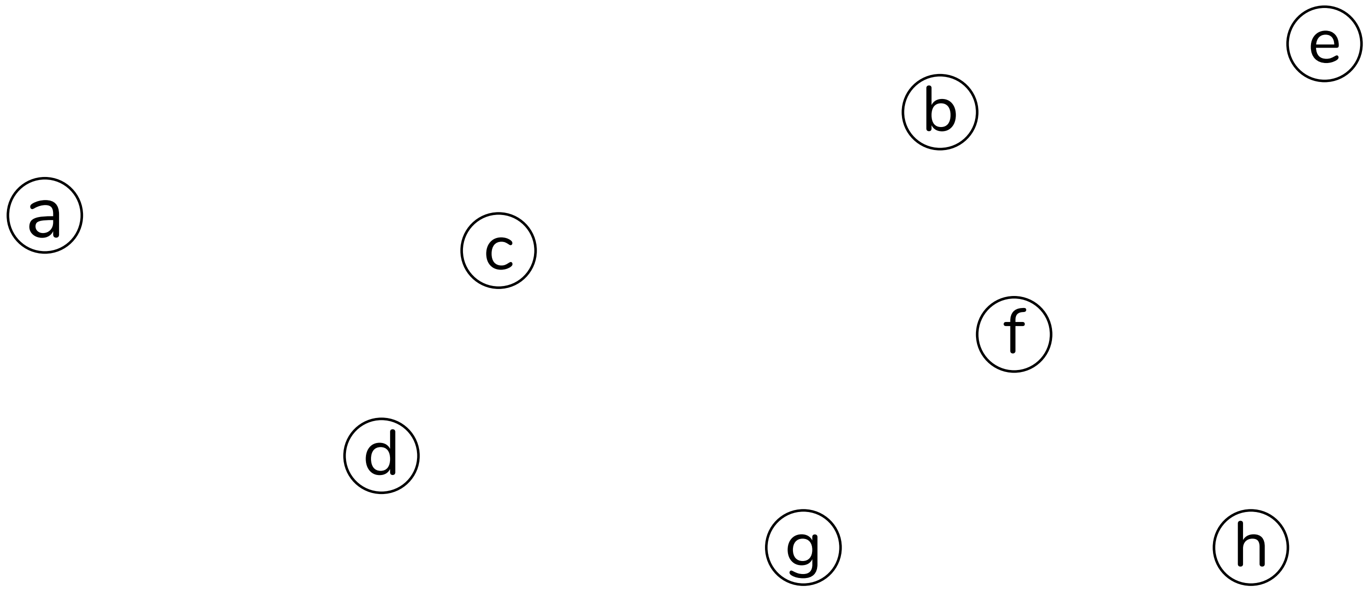
The Traveling Salesperson Problem

Given: n points in R^2

point $a = (a_x, a_y)$, distance $d(a,b) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}$



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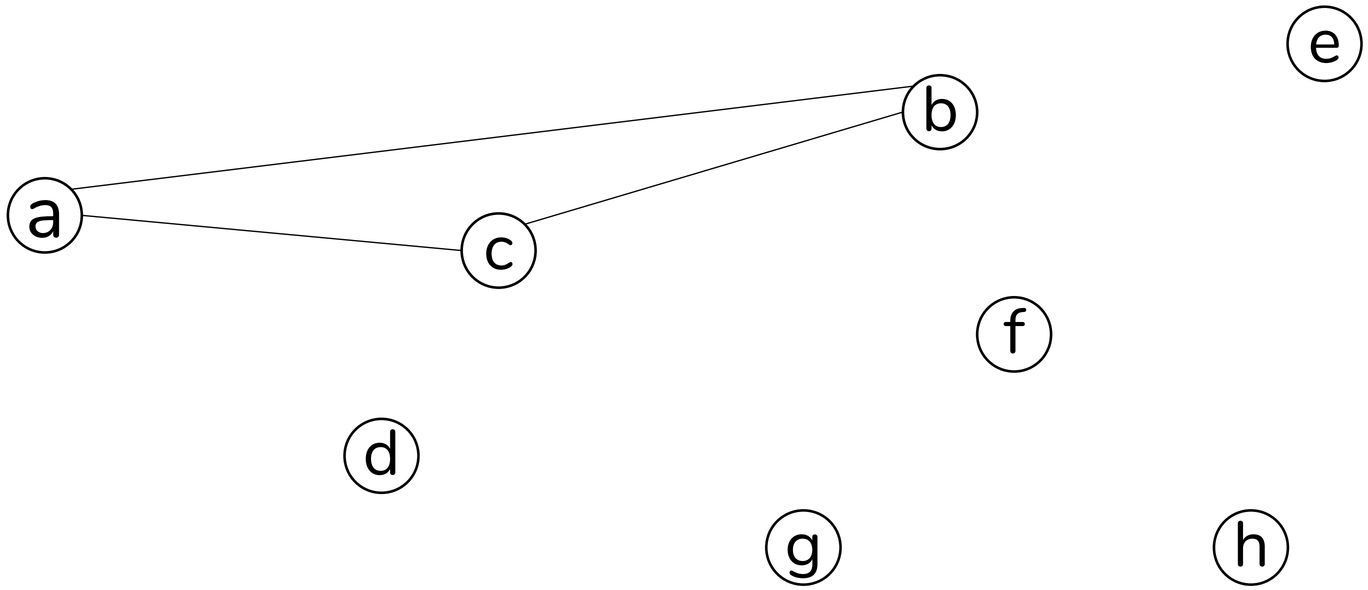

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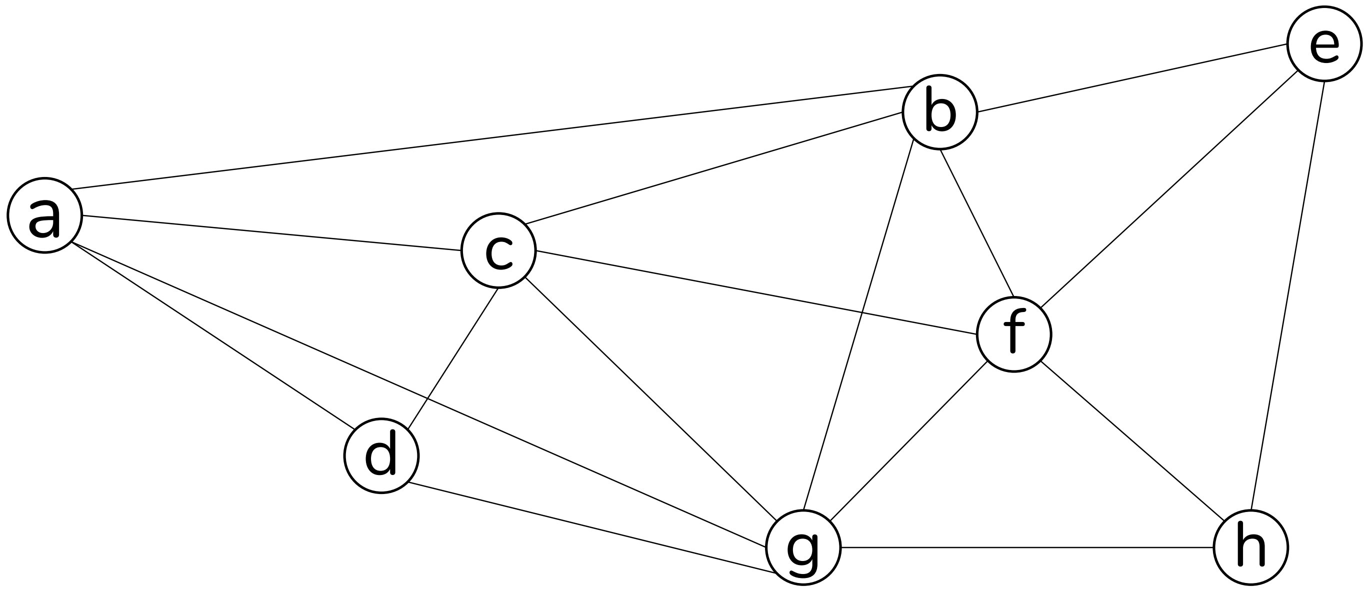

(complete graph with edge lengths)

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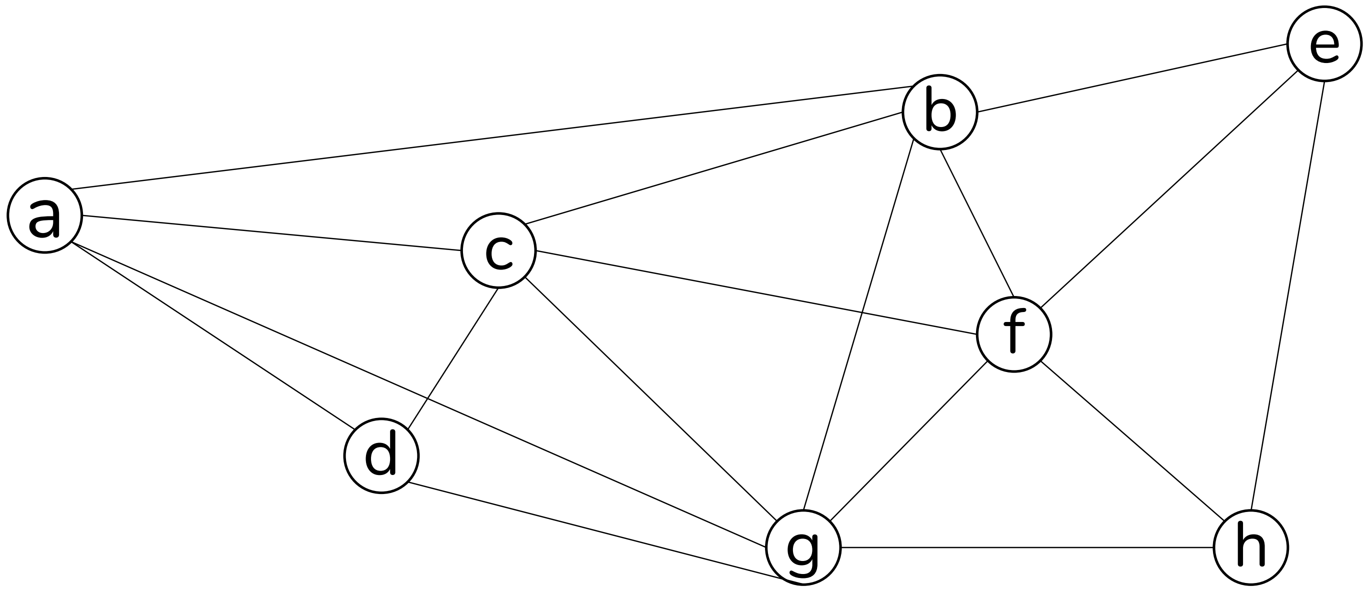
TSP: find shortest possible cycle that visits all points.



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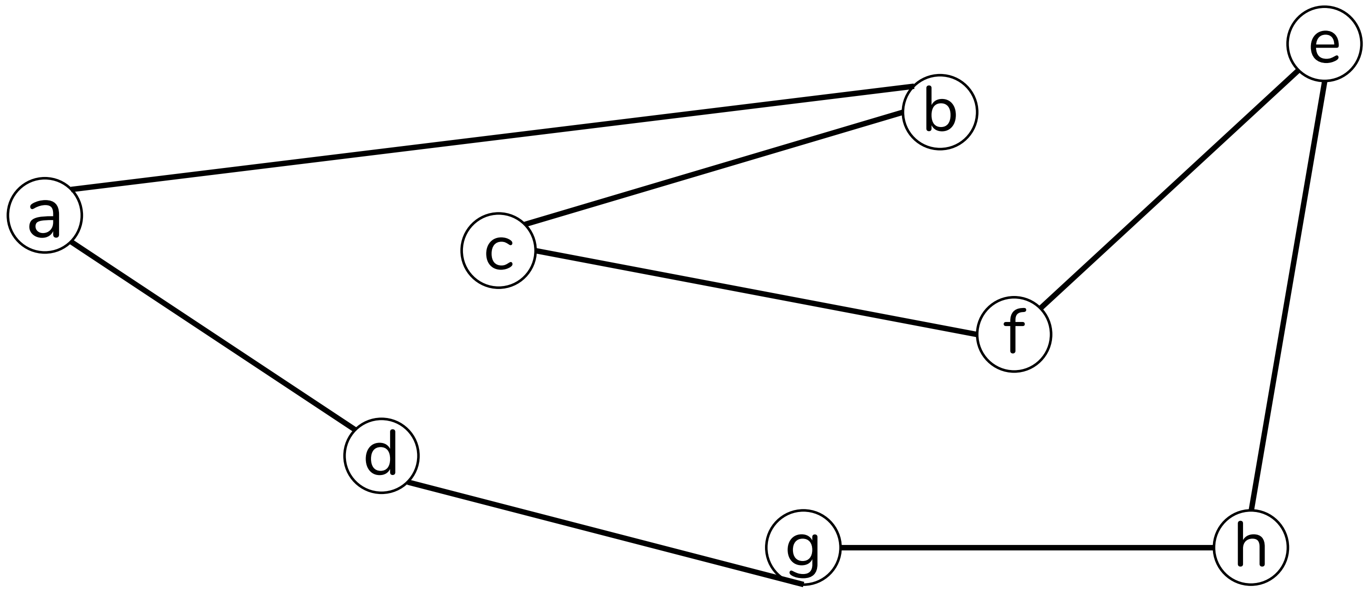

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Trivial algorithm:

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Trivial algorithm: Each cycle corresponds to a permutation of the vertices

Try all $n! \sim 2^{n \log n}$ permutations

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Better algorithms? Can do $O(2^n)$ by a dynamic program (Held-Karp algorithm)

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$\boxed{\text{TSP is NP-hard}}$

(what does this mean?)

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There probably isn't a polynomial-time algorithm for this problem.

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Instead of an optimal solution, we look for a near-optimal solution.

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For any instance I of n points in R^2,
let $OPT(I)$ denote the length of the shortest possible cycle.

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A 2-approximate algorithm gives a cycle of length at most $2 \times OPT(I)$.

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We will give a 1.5-approximate algorithm for TSP.

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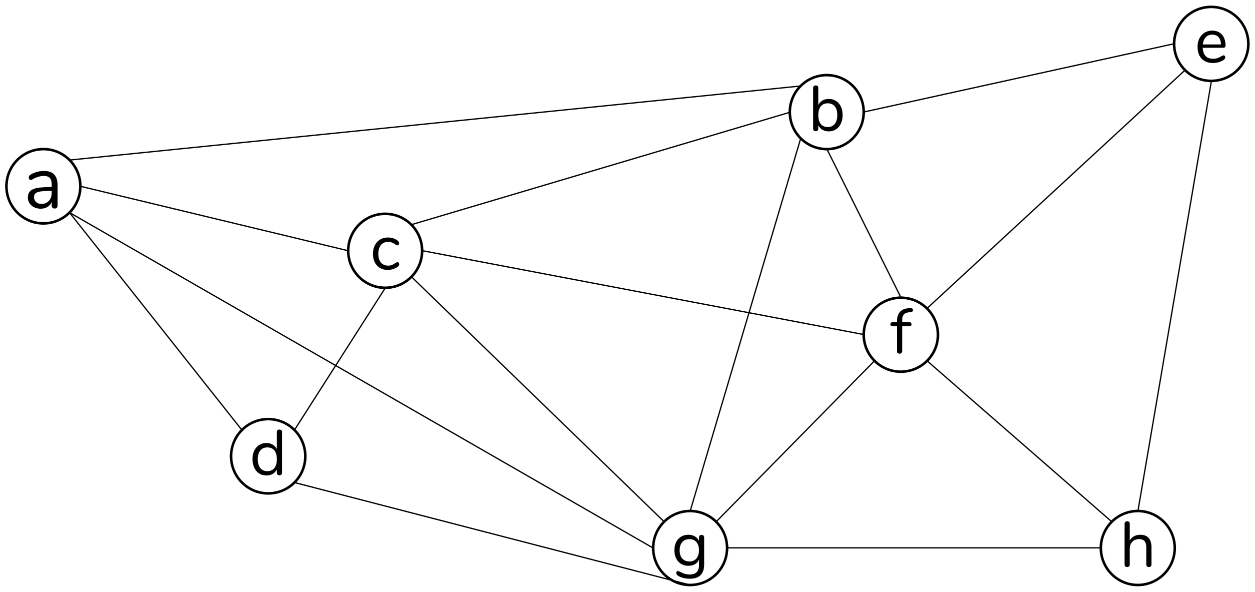

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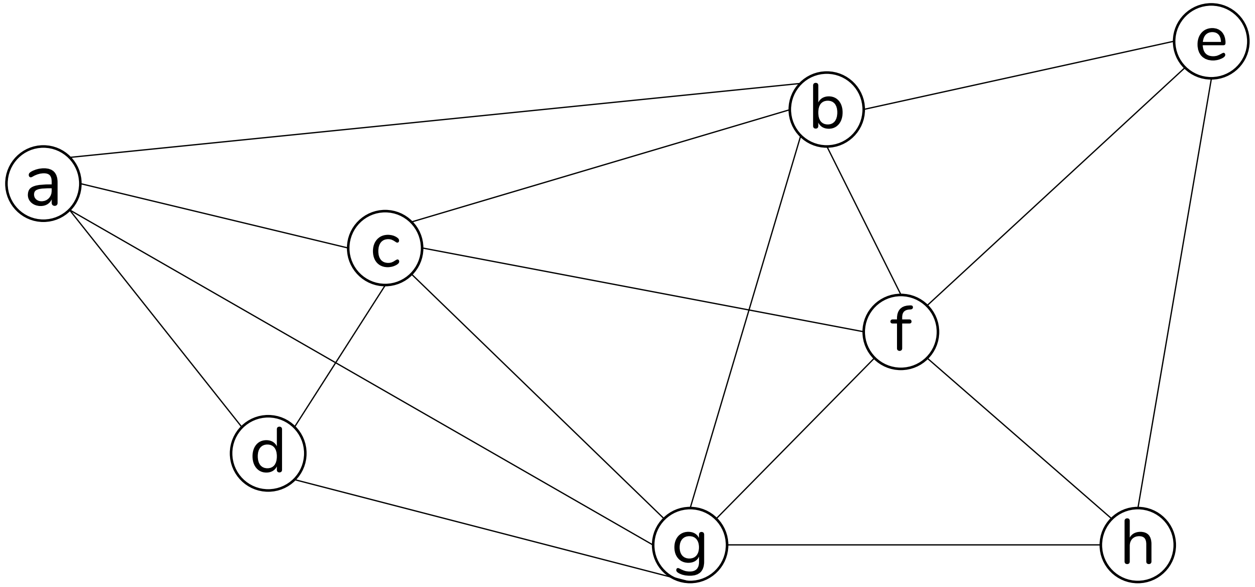
graph



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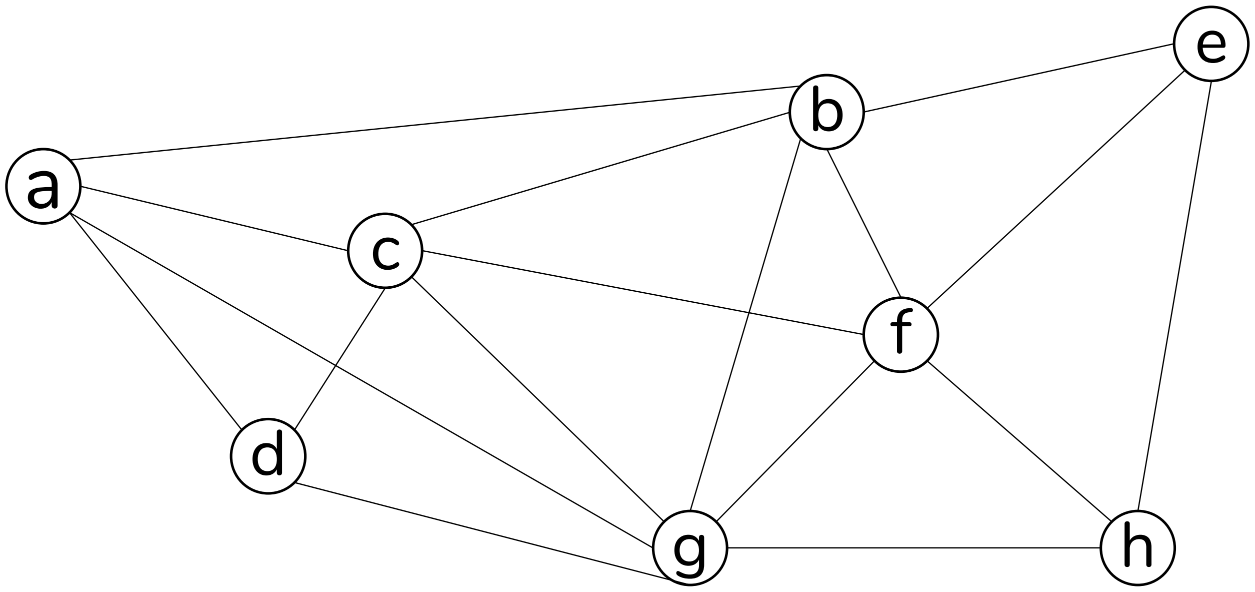
This is a minimum spanning tree!



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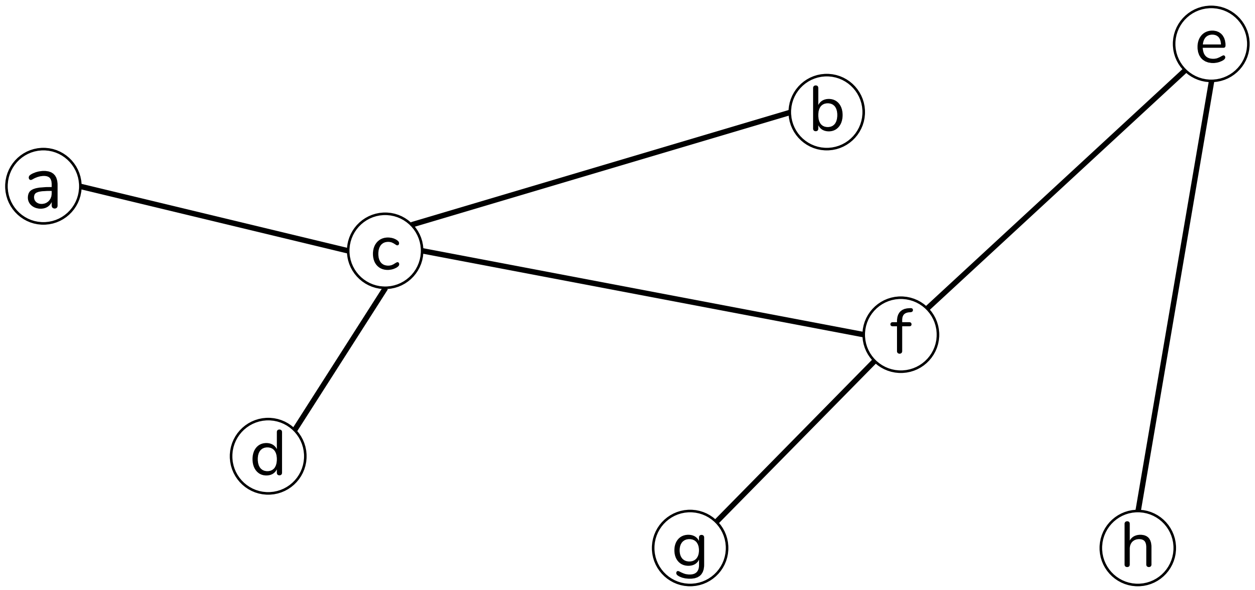

Claim: length of MST $\leq$ length of shortest cycle that visits all pts.

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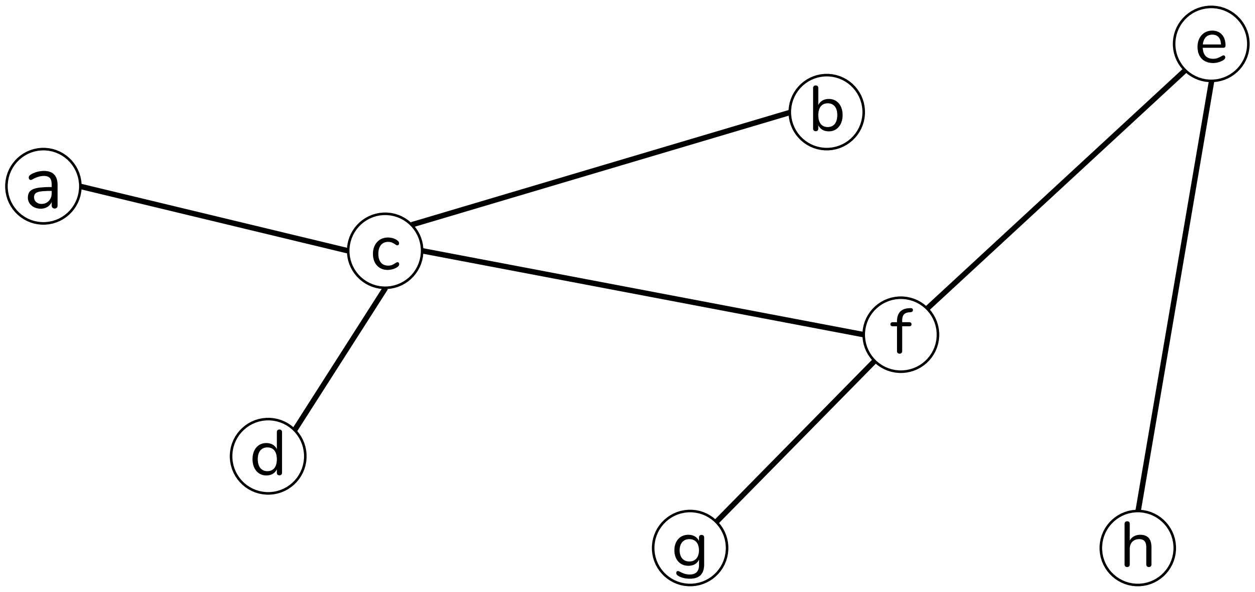

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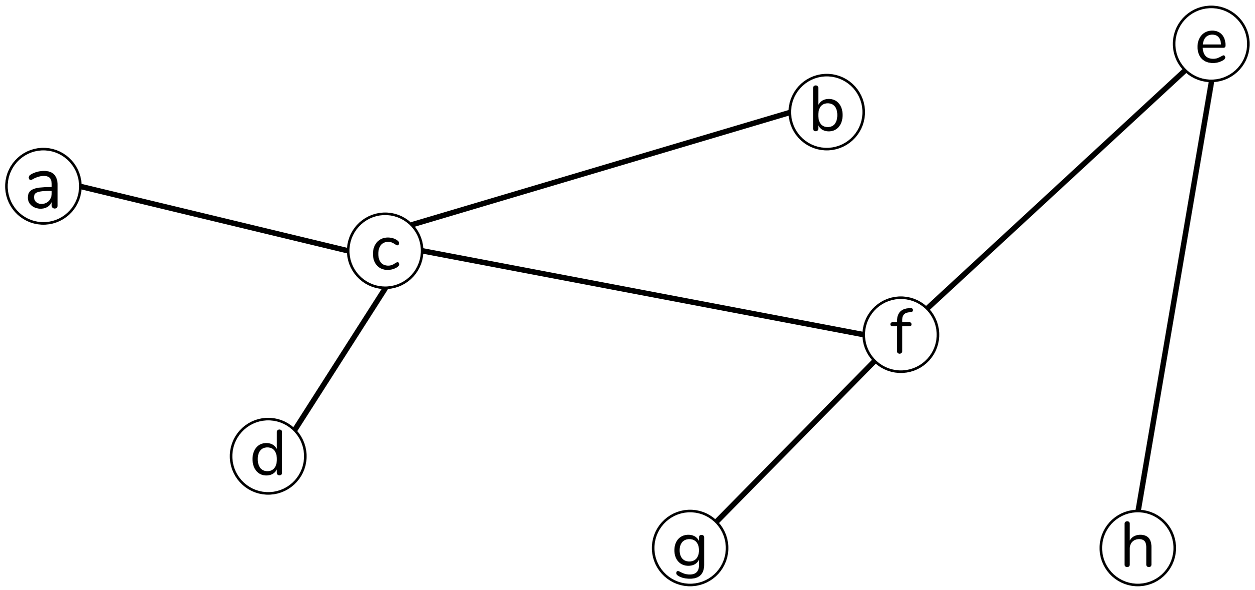

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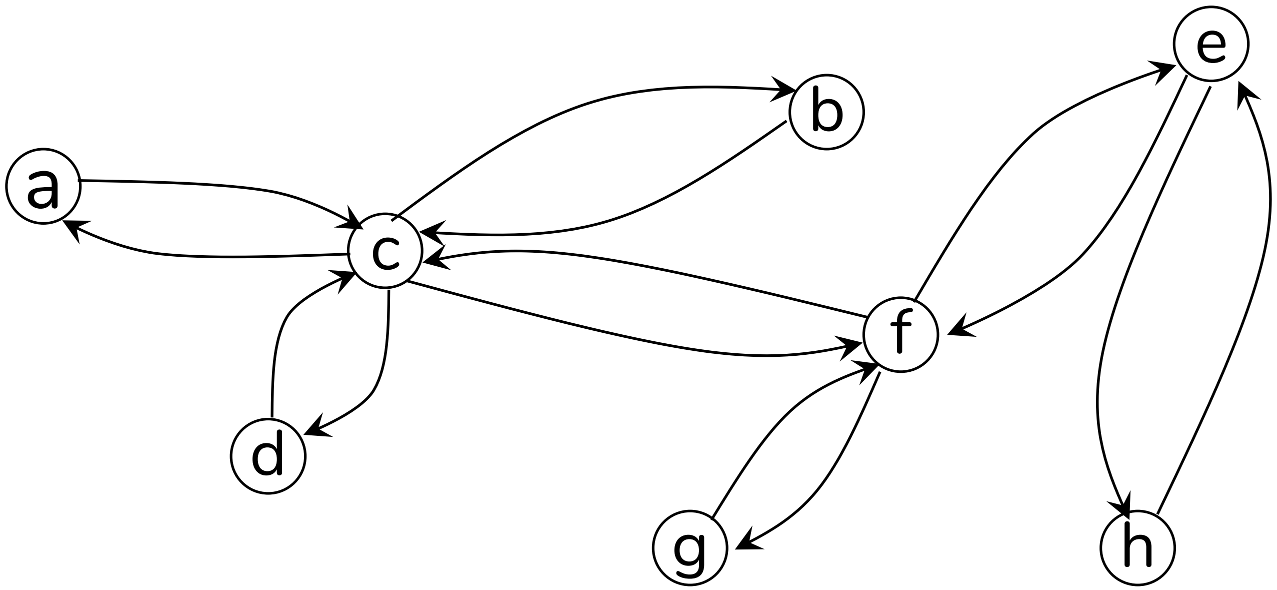

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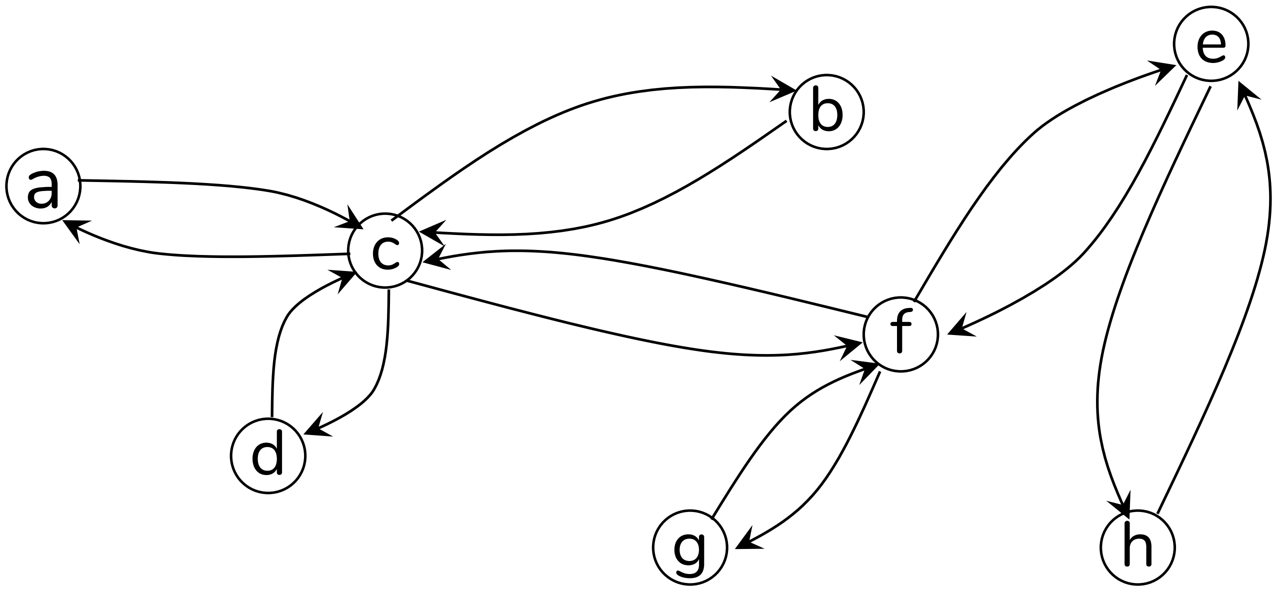

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By doubling edges of MST, can get a walk of length 2 x MST $\leq$ 2 x OPT

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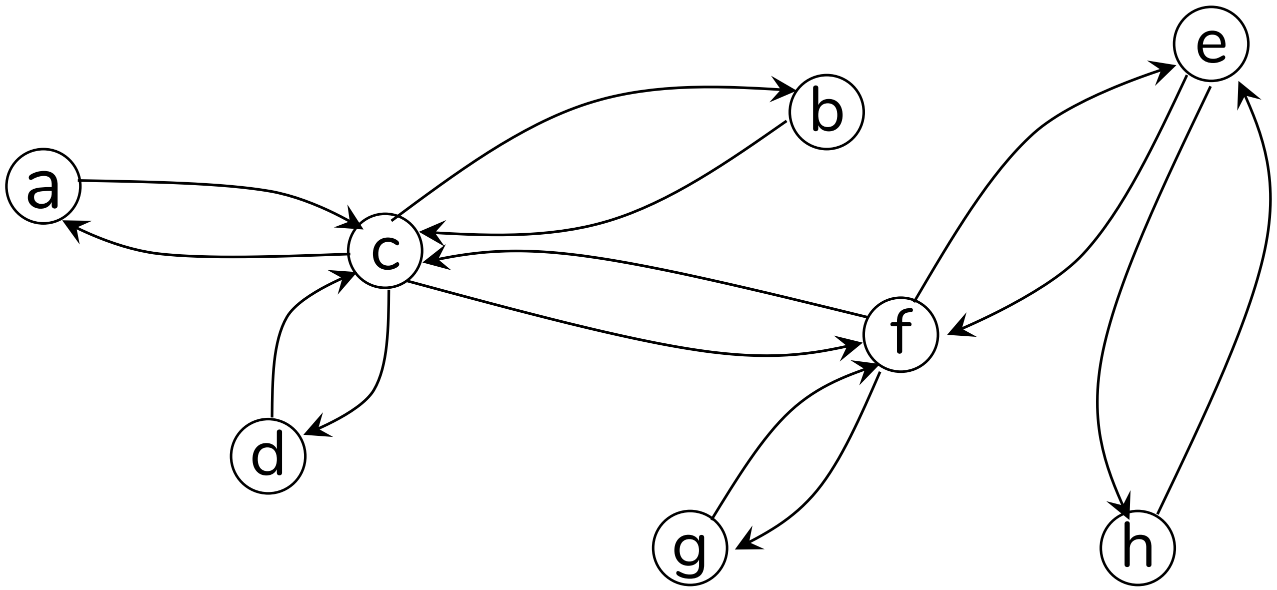

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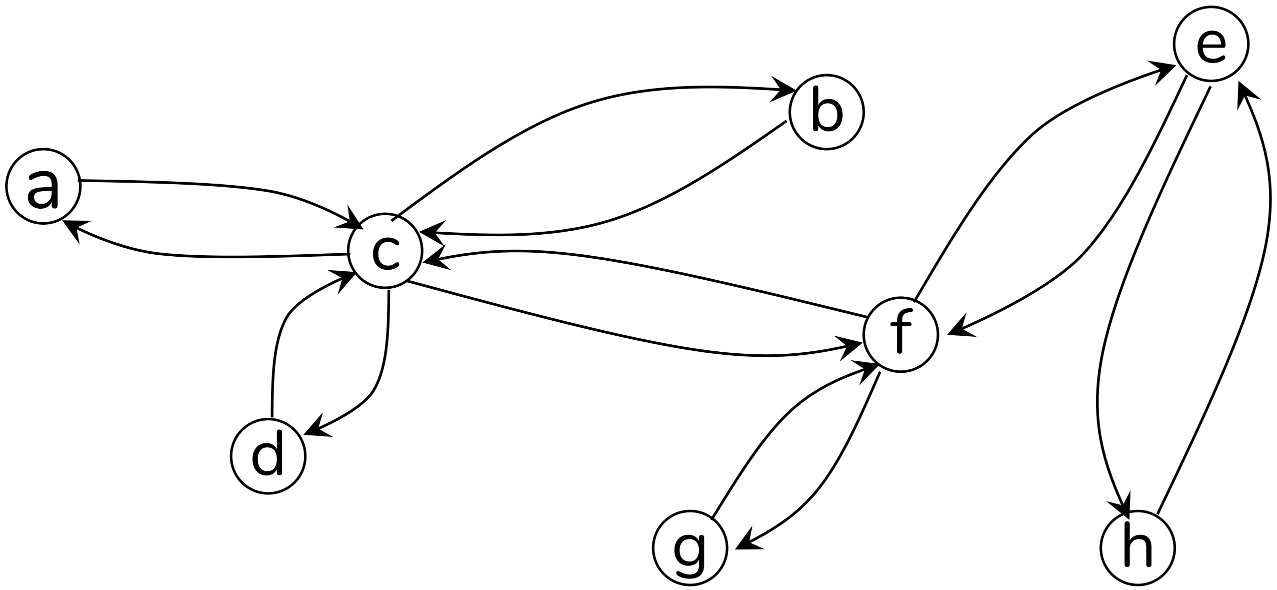

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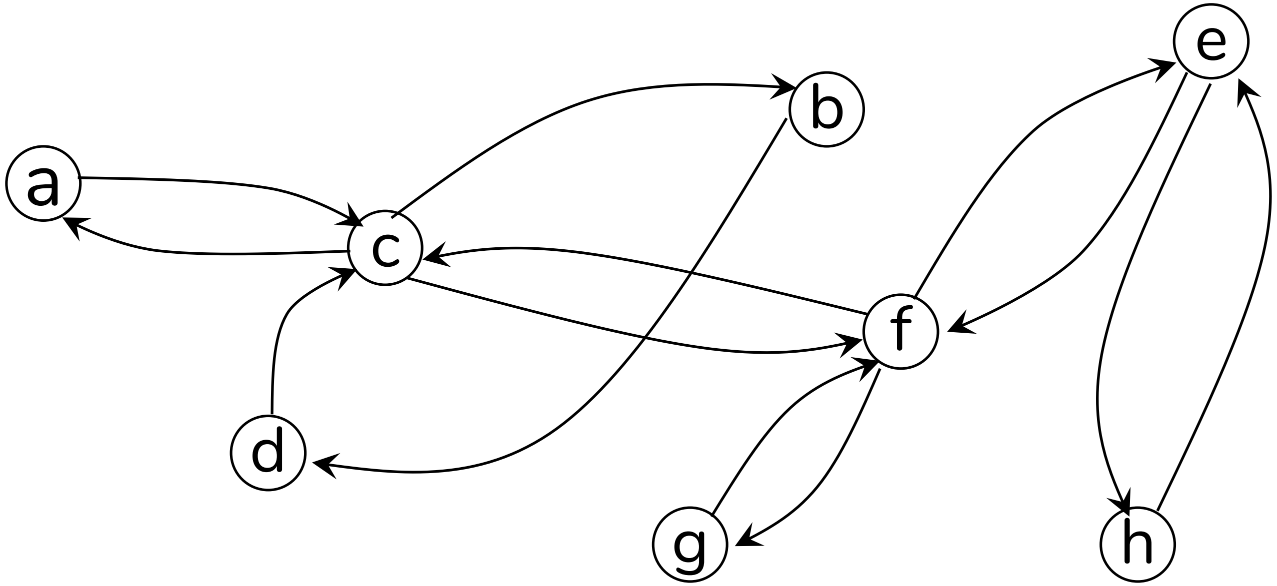

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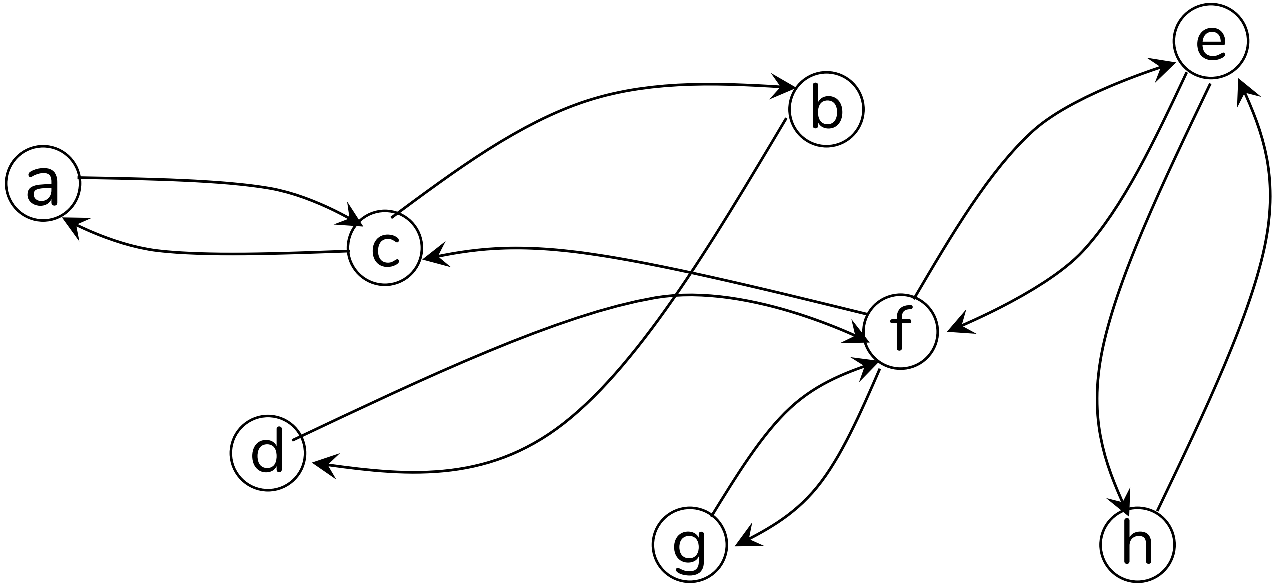

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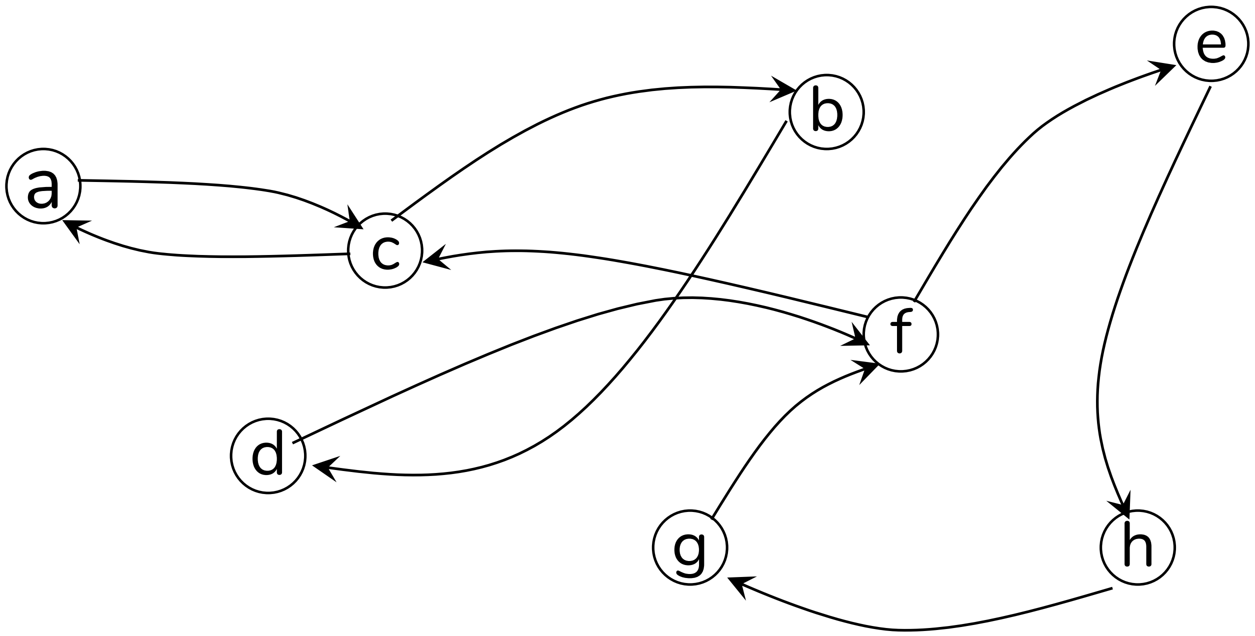

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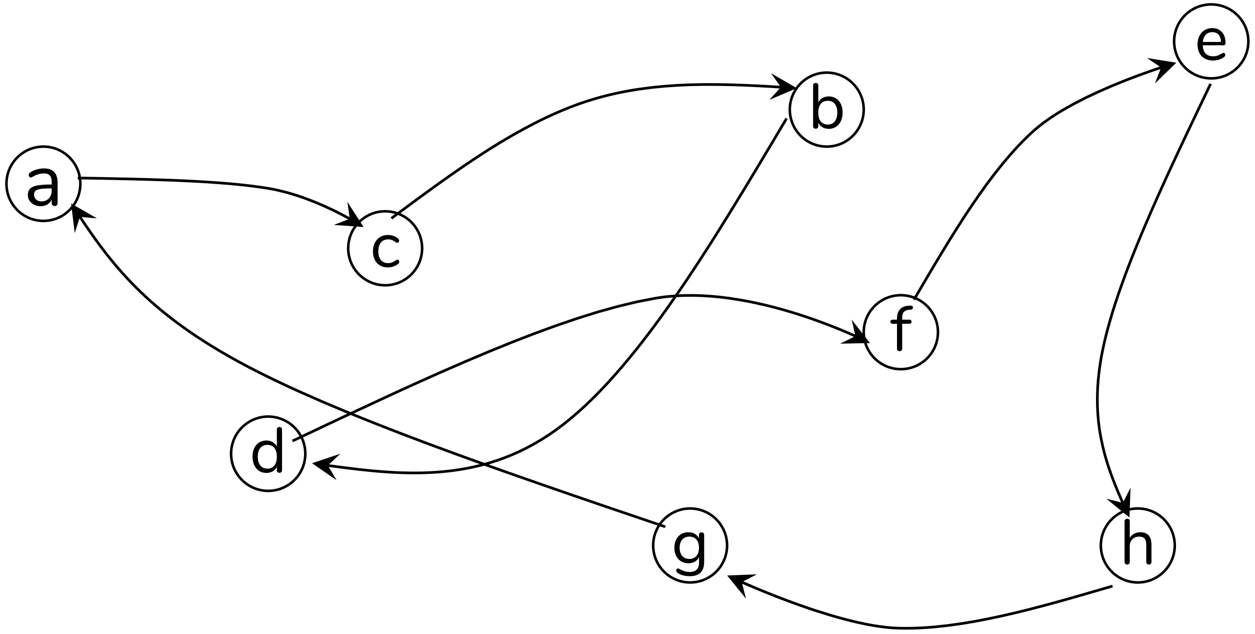

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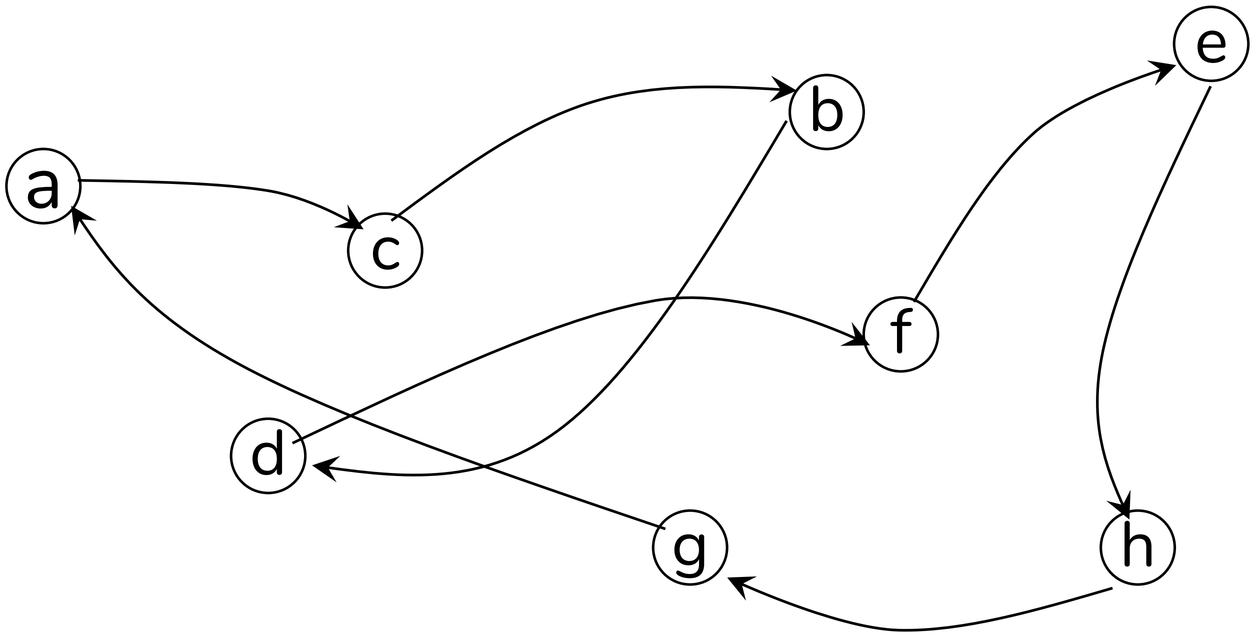

This gives us a 2-approximation algorithm for TSP.

Can short-circuit walk to get a cycle of length $\leq$ 2 x MST $\leq$ 2 x OPT

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Given: n points in R^2 find shortest possible cycle that visits all points.

A 2-approximation algorithm:

1. Find MST (length ≤ OPT)
2. Double edges, find Eulerian walk (length ≤ 2 x OPT)
3. Short-circuit edges to get cycle of length ≤ 2 x OPT

(Remainder on board)